Today's goal is to get comfortable with the basic vocabulary of surfaces by drawing. Don't worry about formal definitions; focus on the pictures.

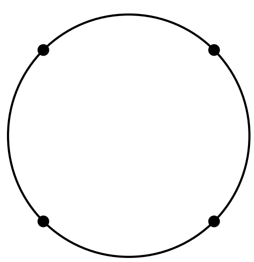
Arcs

Definition: An **arc** γ on the marked surface $(\mathbf{S}, \mathbf{M})$ is a curve on $\mathbf{S}$ such that

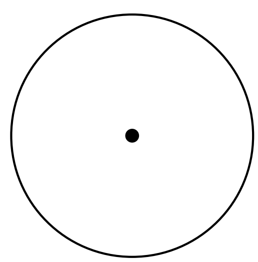
- The endpoints of γ are in $\mathbf{M}$.
- Only the endpoints of γ are allowed to touch the boundary of $\mathbf{S}$.
- γ does not cut out an unpunctured monogon or unpunctured bigon.
- γ cannot cross itself except at the endpoints.

Problems:

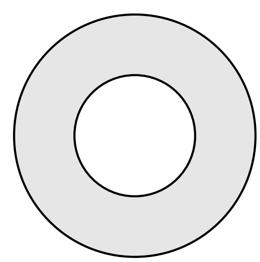
1. On the disk below, draw an arc connecting two different marked points on the boundary. (Careful! Make sure you don't cut out an unpunctured bigon!)
2. On the punctured disk, draw an arc connecting a marked point on the boundary to the puncture.
3. Add some marked points to the annulus to make it into a marked surface. Draw some arcs on it!



(A) Disk with 4 points

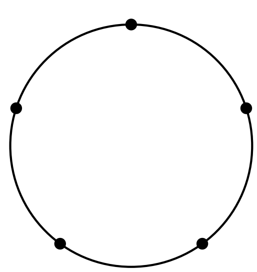


(B) Once-punctured disk

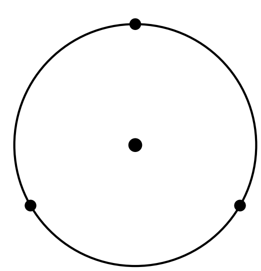


(C) Annulus

4. On either surface below, draw a curve that is **NOT** a valid arc and briefly say why.



Disk with 5 marked points

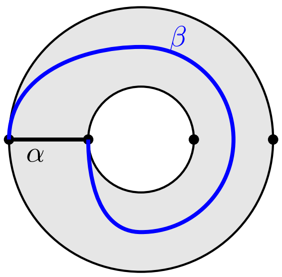


Punctured disk

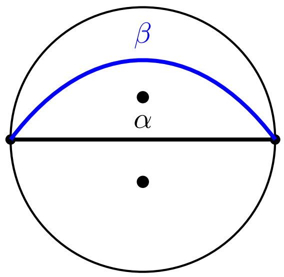
5. Explain why the disk with 4 marked points can be thought of as a quadrilateral. Explain why the disk with 5 marked points can be thought of as a pentagon. Can this be generalized?

Isotopy and Compatibility

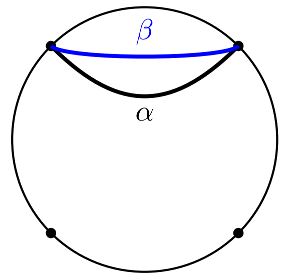
1. For each pair of arcs (α and β) below, answer two questions:
   (a) Are they **isotopic**? (Can you wiggle arc α to look exactly like arc β?)
   (b) Are they **compatible**? (Can they be drawn on the surface without crossing?)



Isotopic? Yes / No
Compatible? Yes / No



Isotopic? Yes / No
Compatible? Yes / No



Isotopic? Yes / No
Compatible? Yes / No

2. For which of the diagrams above do there exist infinitely many arcs that are not isotopic to each other?

Triangulations and Euler Characteristic

A fundamental question in topology is: what properties of a surface stay the same no matter how we draw on it? Let's find out by triangulating two key surfaces.

The Sphere

(For this exercise, you can draw on a circle, which represents a flattened map of a sphere.)

1. **Trial 1:** Choose **at least 3 marked points** and draw a triangulation on the sphere.

2. Count your vertices (V), edges (E), and faces (F), then calculate the Euler Characteristic, $\chi = V - E + F$.

3. **Trial 2:** Repeat the process, this time using a **different number of marked points** or a different triangulation.

4. Compare your final value of χ with a neighbor who used a different triangulation. What do you notice?

The Torus

1. Choose **at least 3 marked points** and draw a triangulation on the torus.

2. Count your vertices (V), edges (E), and faces (F), then calculate the Euler Characteristic, $\chi = V - E + F$.

3. Compare your final value of χ with a neighbor who used a different triangulation. What do you notice?

Other questions to think about

1. Why do you think that the Euler characteristic doesn't depend on the choice of triangulation of the surface? Can you give an argument? (Proving this rigorously is very subtle and is a major topic in introductory algebraic topology!)

2. Try computing the Euler characteristic of the annulus. Notice that this tells you that even though the annulus shares Euler characteristic with another type of surface, the other surface is quite different. This because the annulus has **two distinct boundary components** (the "inner" and "outer" boundaries).

3. Perhaps somewhat counterintuitively there exist surfaces for which there are infinitely many triangulations. See if you can come up with an example! (*Hint*: Try taking a polygon and puncturing it a couple times).

4. A fundamental theorem of algebraic topology is the **classification of compact surfaces**. It says that topologically, any oriented surface **(without boundary — surfaces with boundary, like the annulus and disks, are considered separately)** can be thought of as a sphere, a torus, or a bunch of tori glued together. Why do you think this is?